

# A SHORT NOTE ON THE ORDER OF THE ZHANG-LIU MATRICES OVER ARBITRARY FIELDS

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**ABSTRACT.** We give necessary and sufficient conditions for the Zhang-Liu matrices to be diagonalizable over arbitrary fields and provide the eigen-decomposition when it is possible. We use this result to calculate the order of these matrices over any arbitrary field. This generalizes a result of the second author.

## 1. INTRODUCTION

The various generalizations of the pascal matrices have been an active subject of research since at least the early 1990's (the interested reader can see [2], [6], [9] for background on some of the more well studied variations). The study of the order of these matrices over finite rings, however, is much more recent and can be traced to Deveci and Karaduman [4]. In their article, an explicit function which calculates the order of the generalized Pascal matrix of the first kind over the ring  $\mathbb{Z}/n\mathbb{Z}$  was given.

These matrices are defined in the following way:

$$P_1(y) = (p)_{ij} = \begin{cases} y^{j-i} \binom{j-1}{i-1} & \text{if } j \geq i \\ 0 & \text{otherwise.} \end{cases}$$

Where  $y \in \mathbb{Z}/n\mathbb{Z}$ . However, analogous results for the symmetric Pascal matrices and the generalized Pascal matrices of the second kind over finite fields have remained elusive (despite the partial results in [8], [4], [5] and [3]). The Pascal matrices of the second kind are the square matrices defined as follows:

$$P_2(x) = (p_2)_{ij} = \begin{cases} x^{j+i-2} \binom{j-1}{i-1} & \text{if } j \geq i \\ 0 & \text{otherwise.} \end{cases}$$

where  $x \in \mathbb{F}^\times$ .

One type of matrix which has yet to be examined in this context are the Zhang-Liu matrices. These were introduced in 1998 in [8]. The utility of these matrices is that, in essence, they “give” us both generalizations of the Pascal matrix. They are defined as:

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$$Q(y, x) = (\rho)_{ij} = \begin{cases} (y^{j-i})(x^{j+i-2}) \binom{j-1}{i-1} & \text{if } j \geq i \\ 0 & \text{otherwise.} \end{cases}$$

where  $x$  is in the multiplicative group of a field, and  $y$  is any element of the field.

In this note, we give necessary and sufficient conditions for  $Q(y, x)$  to be diagonalizable over a given field. We also determine the exact order of  $Q(y, x)$  as a function of  $y$  and  $x$ .

Throughout this note,  $p$  will denote a prime, we will assume that  $y \in \mathbb{F}$  is a field with multiplicative group  $\mathbb{F}^\times$  and that  $x \in \mathbb{F}^\times$ . We will also assume that all matrices are square of dimension  $n \geq 2$  and we denote  $|a|$  as the multiplicative order of an element  $a$  of a field. Also, we operate under the convention of [6] and [2] that  $0^0 = 1$ .

## 2. RESULTS

The desired theorems will stem from the following factorization.

**Theorem 1.** *If  $x \notin \{1, -1\}$  and  $z = \frac{yx}{x^2-1}$  then the following equation holds:*

$$Q(y, x) = P_1(z)D(x^2)P_1(-z),$$

where  $D(\alpha)$  is the diagonal matrix consisting of nonzero entries  $(d)_{i,i} = (\alpha)^{i-1}$  where  $\alpha \in \mathbb{F}$ .

*Proof.* Given that  $x \notin \{1, -1\}$ , then  $x^2 - 1 \neq 0$  and so  $\frac{yx}{x^2-1}$  exists. Since all the matrices in the product are upper triangular, we will concern ourselves only with the case when  $j \geq i$ . We denote the  $i, j$  entry of  $P_1(z)$  by  $p_{i,j}$  and the  $i, j$  entry of  $P_1(-z)$  by  $p'_{i,j}$ . Then, letting  $b_{i,j}$  be the  $i, j$  entry of the matrix  $P(z)D(x^2)P(-z)$  we see that by defining

$$a_{i,j} := \sum_{k=1}^n d_{i,k} p'_{k,j} = (x^2)^{i-1} \left( \frac{-xy}{x^2-1} \right)^{j-i} \binom{j-1}{i-1}$$

we can conclude that

$$\begin{aligned} b_{i,j} &= \sum_{k=1}^n p_{i,k} a_{k,j} = \sum_{k=i}^j \left( \frac{xy}{x^2-1} \right)^{k-i} \binom{k-1}{i-1} (x^2)^{k-1} \left( \frac{-xy}{x^2-1} \right)^{j-k} \binom{j-1}{k-1} \\ &= \sum_{k=i}^j (-1)^{j-k} \left( \frac{xy}{x^2-1} \right)^{j-i} \binom{j-1}{k-1} \binom{k-1}{i-1} (x^2)^{k-1} \\ &= \left( \frac{xy}{x^2-1} \right)^{j-i} \sum_{k=i}^j (-1)^{j-k} \binom{j-1}{i-1} \binom{j-i}{k-i} (x^2)^{k-1} \\ &= \left( \frac{xy}{x^2-1} \right)^{j-i} \binom{j-1}{i-1} \sum_{k=i}^j (-1)^{j-k} \binom{j-i}{k-i} (x^2)^{k-1} \\ &= \left( \frac{xy}{x^2-1} \right)^{j-i} \binom{j-1}{i-1} \sum_{k=0}^{j-i} \binom{j-i}{k} (-1)^{j-i-k} (x^2)^{k+i-1} \end{aligned}$$

$$\begin{aligned}
&= \left( \frac{xy}{x^2-1} \right)^{j-i} \binom{j-1}{i-1} (x^2)^{i-1} \sum_{k=0}^{j-i} \binom{j-i}{k} (-1)^{j-i-k} (x^2)^k \\
&= \left( \frac{xy}{x^2-1} \right)^{j-i} \binom{j-1}{i-1} (x^2)^{i-1} (x^2-1)^{j-i} \\
&= (xy)^{j-i} \binom{j-1}{i-1} x^{j+i-2}. \\
&= (y^{j-i})(x^{j+i-2}) \binom{j-1}{i-1}
\end{aligned}$$

Proving that  $b_{i,j} = \rho_{i,j}$ . □

This factorization leads to the following observation regarding the eigenvectors of  $Q(y, x)$ .

**Corollary 2.** *If  $x \notin \{1, -1\}$ , then the eigenvectors of  $Q(y, x)$  are the columns of  $P(\frac{yx}{x^2-1})$ .*

*Proof.* It is simple to verify that  $P_1(z)^{-1} = P_1(-z)$  (in the case of the real numbers, this has been done explicitly in [2], and in fact, their proof holds for arbitrary commutative rings with identity). Thus the factorization provided by Theorem 6 is the eigen-decomposition of  $Q(y, x)$ . □

Combining this, with the observation that if  $x \in \{-1, 1\}$  and  $y \neq 0$  that  $Q(y, x)$  is deficient leads to the following desired theorem.

**Theorem 3.** *The Zhang-Liu matrix  $Q(y, x)$  is diagonalizable if and only if  $x \notin \{-1, 1\}$  or  $y = 0$ .*

The next corollary gives us the order of  $Q(y, x)$ .

**Corollary 4.** *Let  $k = |x^2|$ ,*

- (1) *If  $x \notin \{1, -1\}$  then  $|Q(x)|$  is  $k$ .*
- (2) *If  $x \in \{1, -1\}$  and the characteristic of  $F$  is  $q \neq 0$  then  $|Q(x)| = q$ .*
- (3) *If  $x \in \{1, -1\}$  and the characteristic of  $F$  is 0 then  $|Q(x)| = \infty$ .*

*Proof.* If  $x \notin \{1, -1\}$  then by Theorem 1,  $Q(y, x)$  is diagonalizable over  $\mathbb{F}$  with all eigenvalues a power of  $x^2$  thus  $|Q(y, x)| = |x^2|$ .

To complete proof, note that  $Q(y, 1) = P(y)$ , thus  $Q(1, 1) = P(1)$  which by [4] is of order  $p$  if  $\mathbb{F}$  is of characteristic  $p$ . A similar argument holds if  $x = -1$ . □

We remark that this last corollary is a vast generalization of Theorem 2.4 of [5] which says that if  $F$  is a field with  $p$  elements and  $|P_2(x)| < p$ , then  $|P_2(x)| = |x^2|$ .

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